

3. THE INFINITE

§3.1. Is Infinite Knowledge Possible?

We're told that God is omniscient. He knows all. We don't. We have finite brains so we can't know infinitely many different things. Or can we?

The finite brain has a large, but finite, memory capacity. So common sense tells us that there's a finite amount of information it can contain. Yet in a certain sense we can know infinitely many facts. (Of course this falls far short of omniscience!)

Suppose I am in love with a woman, living in another city. I write and tell her that I love her. At the time of writing she doesn't know that. But the next day, when she receives the love letter, she knows. She knows that I love her but I don't know that she knows. Not until she writes back. When I get her reply, I know that she knows I love her, but she doesn't yet know this. She must wait for my reply to find out.

There are infinitely many facts of the type: "know that she knows that I know that she knows" After a few letters it may not be of very great interest to me to distinguish between these successive layers. But logically they're separate facts because they become true on successive days.

Now although there are infinitely many distinct facts here, at any time only a finite number will be known to me. Every second day when I open the next letter from my beloved I have a new fact to add to my collection, but only ever a finite number at any stage.

But suppose that instead I was in her presence, looking lovingly into her eyes. At the moment I issue the words, “I love you”, all of the “I know that she knows .” facts are instantly known to me. It’s not that the information travels back and forth rapidly at the speed of light. I don’t need to wait for her eyes to light up. I can infer that because she is actually in my presence, he has heard what I said. And equally well she can deduce that because I am standing there, I will know that she knows. Infinitely many distinct facts are knowable in an instant!

I know that 2 is bigger than 1. That’s one fact. And 3 is bigger than 2 – a second fact. And 4 is bigger than 3, and 5 exceeds 4, and so on. There are infinitely many facts like these, all of which I know. And the reason why I know them is that I have a rule: “one more than any number exceeds that number”, or “ $n + 1 > n$ ”.

The only way a finite mind can know an infinite number of facts is to know a rule that will generate them. We will only ever use that rule a finite number of times, but the set of potential instances is infinite. In this sense, a finite mind can have infinite capability.

§3.2. What Does ‘Infinite’ Mean?

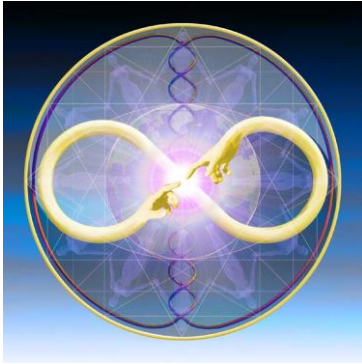
It’s an amazing thing that the finite human mind ever conceived of the idea of the infinite. As amazing as if a race of natives living on a small island which had always been completely cut off from the rest of the world, had a concept of other lands across the sea.

What’s even more amazing is that the finite human mind has to some extent been able to explore the infinite and has discovered detailed facts about infinity. As amazing as if that isolated island race had detailed knowledge of the cathedrals of Europe.

Yet that is the case. Mortal man has notions of immortality, earth-bound man conceived of other worlds long before space travel began, and finite man has had a word for “infinity”. But what exactly is infinity?

I once asked a group of students what they thought ‘infinity’ meant. Some said, “it’s the biggest number there is”. Others said, “it’s something you can approach but never reach.” Yet others said, “the ultimate”.

All of these answers have captured a little of the mystery of the infinite but they’re notions far too vague on which to build any knowledge.



“Infinity is the biggest number there is.” Well of course there is no biggest number in the sense of a number in the sequence 1, 2, 3, ... So we’ll have to invent a new number that goes after all the others:

1, 2, 3,, ∞

How does that sound? Any set or collection that isn’t finite is infinite, and the number of elements in an infinite set is denoted by ∞ .

That’s a perfectly good state of affairs if we don’t want to be discriminating about the infinite. We’re saying that all infinite sets are to be regarded as having equivalent size – there’s only one infinity.

There’s nothing wrong with this – except that it’s a bit like the tribe of Tasmanian aborigines who are supposed to have had no words for numbers after three. Counting in their language went “one”, “two”, “three”, “many”. Any more than three is a crowd. That certainly keeps arithmetic very simple! “Two plus two is many”.

Anthropologists have dismissed this story as false and indeed we’re discovering that aboriginal culture was

rather more sophisticated than we thought. Elsewhere I describe how the kinship rules of one tribe anticipated the discovery of group theory.

Georg Cantor discovered, in the latter part of the nineteenth century, that it's possible to distinguish between different sizes of infinity in a very natural way and this has proved a useful tool in both mathematics and computer science.

Not just one infinity, but many. Now if you're hearing about this for the first time you're perhaps a little sceptical. It's quite a radical idea, even though it's been around for over a hundred years. At least keep an open mind on the question. Simply to automatically lump all infinite sets under the one heading is to make up your mind in advance.

Well if we're to proceed and to ask the question whether or not all infinite sets have the same size, we need to develop some concept of size or "same size". In chapter 1 we saw that we could define two sets of things as having the same size if they can be paired off exactly.

Two sets have the same size if we can pair the elements of one exactly with the elements of the other.

§3.3. Counting Couples

One of the greatest hindrances to social harmony in a society is held to be an imbalance between the sexes. Nowhere is this more evident than at a formal dance.

Now I know that the problem of insufficient men at a dance has often been overcome by women dancing with women, and in modern times the necessity or desirability of dancing with a member of the opposite sex, or indeed having a partner at all, has been called into question. But for the old-fashioned formal balls, for which the Strauss brothers wrote their waltzes and polkas, it was taken as an axiom that dancers were couples and each couple came from opposite sides of the biological tracks.

Imagine then that you're in Vienna at a ball and that you cast your eyes around the many dancing couples. You notice that nobody is sitting out – all are dancing. You'd be justified in concluding that the number of men was the same as the number of women, that is, if you exclude yourself. If, on the other hand, there were a few female wallflowers, and no men, you'd conclude that there were more women than men.

These conclusions would have been reached without counting the men and counting the women and then doing the necessary arithmetic comparison.

Are there more left legs or right legs on the dance floor? A brief inspection reveals no one-legged dancers hobbling on crutches, so the number of left legs is the same as the number of right legs. Again, no counting was involved. Just the realisation that left legs are paired with right legs, just as male dancers are linked to their female partners.



§3.4. The Biggest Number There Is

We tend to think of counting as the most basic of all mathematical activities. Yet more primitive still is the notion of one-to-one correspondences, or pairings.

When, as kindergarten children, we counted out loud as we pointed to each object in turn, we were setting up a one-to-one correspondence between the things we were counting and a certain set of counting numbers. We pointed to a yellow duckling and said “one”. The next one was called “two”. We may have thought, at one stage in our conceptual development, that we were giving names to the fluffy creatures.

Gradually it would have dawned on us that these ‘names’ have nothing to do with what we were counting as we abstracted the concept of number from the things

themselves. Soon we felt very proud that we could count to a hundred and beyond. As we learnt to write down longer and longer numbers we began to realise that there was no end in sight. We might not have known what words to use after ‘trillions’ and ‘quadrillions of quadrillions’ but we knew that we could keep adding zeros to make larger and larger numbers.

Big numbers fascinate little children and a favourite pastime is to think of a description of a bigger number than other children.

“I bet you a trillion, trillion, quadrillion dollars that”

“I bet you all that and a trillion dollars more!”

“Alright, I bet you all the money in the universe.”

“I bet you a hundred times all the money in the universe.”

It was fortunate that none of these childish bets ever had to be paid. The next stage was the concept of ‘infinity’.

“I bet you infinity dollars.” This was supposed to be a winning move because infinity is the biggest number there is.

“I bet you infinity times infinity dollars!”

§3.5. Dancing To The Music of Schröder and Bernstein

The infinite world is in many ways an extension of the familiar finite world. But in other ways it is quite different. The concept of pairing as the basis for same-number-as works just as well for the infinite as it does for the finite. Where the difference comes is that a finite set gets smaller if you take one thing out. An infinite set does not.

This may seem paradoxical but that's because we're to some extent imprisoned by our experience of the finite world. Remember we've agreed to say that two sets have the same size if they can be paired off exactly with nothing left over.

Dancers in competitions often have numbers pinned to their backs. Imagine a competition with *infinitely* many men and infinitely many women. The dance floor may get a little crowded but with a bit of effort we can read the numbers pinned to them: 1, 2, 3, ... They go on forever.

Number '1' gentleman dances with number '1' lady, '2' dances with '2' and so on. Everyone's happy because the number of men is exactly equal to the number of women. But lady number '1' feels poorly and goes home so number '1' man is without a partner. The

numbers of men and women are no longer the same. Right?

Wrong? All it needs is a little reorganisation. Number '1' man can now dance with number '2' lady. Number '2' man, having lost his partner taps number '3' man on the shoulder and takes over his partner. The dance becomes an 'excuse me' dance as each man changes to the next numbered lady.

In a finite world, the last man misses out. But in an infinite world, there is no last man! Nobody misses out. Soon everyone has his or her new partner and the dancing goes on. Everyone is happy and so we're forced to conclude that the number of men and women has remained the same.

So the fact that there are some men, and no ladies, sitting down not dancing doesn't mean that there are more men than woman. Not at an infinite dance anyway. And if at another dance there are only lady wallflowers it needn't be the case that there are more ladies than men. The numbers may in fact be the same in each case and it may just need a bit of reorganisation of partners to get everyone on the dance floor.

Of course with finite sets of dancers this can't happen. Only women left on the side? There must be fewer men. But with infinitely many it's possible for this apparent disparity to occur with equal numbers.

If lady number '1' had returned after the above excuse-me dance had reorganised the couples, she'd be without a partner, notwithstanding the fact that the numbers of men and women would still be the same. All very strange, but you can't dabble with the infinite without getting a few shocks.

Schröder and Bernstein may sound like a pair of musicians but in fact they were a couple of mathematicians who discovered, and proved, what is known as the Schröder-Bernstein Theorem. If they had been musicians in an infinite ballroom they might have had a conversation like this:

"I say, Schröder, did you notice that in the last dance there were only ladies left over."

"Of course my dear Bernstein. Such poor organisation. In the dance before that there were only men sitting out."

"That surely means that there are equal numbers of ladies and gentlemen."

"Probably, but can you prove it?"

"I'll think about during the next dance."

Schröder and Bernstein did in fact prove this fact, though not while playing at an infinite ball. In less

colourful terms the Schröder-Bernstein Theorem goes something like this (to the tune of “*The Number Rhumba*”)

If all the elements of J can be paired with some of the elements of K, and all the elements of K can be paired with some of the elements of J, then it follows as surely as dead cats have flies that J and K must have the same size.

With finite numbers it’s obvious that if $m \leq n$ and $n \leq m$ then $m = n$. With infinite sets it’s true, but not nearly so obvious. The best we have is $\underline{\underline{\blacktriangleleft}}$.

If S and T are sets, $S \underline{\underline{\blacktriangleleft}} T$ means that all of the elements of S can be paired off with some or all of T.

For example, if $S = \{-2, -4, -6, \dots\}$ and $T = \{1, 2, 3, \dots\}$ then $S \underline{\underline{\blacktriangleleft}} T$ because we can pair of $-n$ with n .

The Schröder-Bernstein Theorem says that: If $S \underline{\underline{\blacktriangleleft}} T$ and $T \underline{\underline{\blacktriangleleft}} S$ then $S \approx T$. It’s a complicated proof. If you’re really interested you can find it in my notes on *Set Theory*.

§3.6. The Infinite Manifesto

A political manifesto is a sequence of statements that are believed to be true about the way society should be run. Of course there will only be finitely many statements in such a manifesto. But consider the following *infinite* manifesto:

INFINITE MANIFESTO



At least one of the following statements is FALSE
At least one of the following statements is FALSE
At least one of the following statements is FALSE
At least one of the following statements is FALSE
At least one of the following statements is FALSE

.....

This appears to be a single statement repeated infinitely many times, but each one refers to a different collection of statements and so they're subtly different. Notice that no statement refers to itself, either directly, or indirectly. There is no circular self-referentiality.

Yet there is a paradox hidden in this seemingly innocuous manifesto. If any one of the statements is FALSE, all the following ones must be TRUE. Yet such a TRUE statement would be followed only by TRUE statements, and so must be FALSE. This is a contradiction.

So all the statements must be TRUE, which is again a contradiction.

Go through the argument slowly a few times until you can see that we cannot assign truth values to these statements in any consistent manner. But note – this example doesn't show that logic is nonsense. It merely shows that the artificially constructed infinite list of, what *appear* to be statements, don't contain any genuine statements at all.

§3.7. The Largest Prime Number

How can you prove that there are infinitely many golden eggs in a magic goose? Just waking up each morning to a new gleaming golden egg is no proof. Perhaps tomorrow there will be none, or the next day.



No, the only way to be certain that there will always be a new egg each morning is to cut the goose open and find out. But, you all know the story!

A prime number is a number bigger than 1 that has no factors other than 1 and itself. The list of prime numbers starts with 2, 3, 5, 7, 11, 13, 17, 19, 23, ... Are there infinitely many prime numbers or is there, somewhere out there, a largest prime number? You can't settle it as easily as showing that there is no biggest number. You can't add 1, or even 2, to a prime number and expect to get a prime number. Prime numbers have fascinated mathematicians for thousands of years because in a certain sense they're as unpredictable as random numbers.

What is certain about them is their statistical distribution. While there's no known formula for the n 'th prime, the probability that a random number of a certain size is prime is known. This probability falls off as the size increases. Primes get rarer and rarer. Could they, in fact, dry up altogether? Euclid proved that they *do* go on forever, even though they become scarcer and scarcer. This is a famous example of a proof by contradiction.

Theorem: There are infinitely many prime numbers.

Proof: Suppose to the contrary that there are finitely many prime numbers.

Multiply them altogether and you get a number which is divisible by them all.

Add one more and you get a number that's not divisible by any of them (a prime number can't divide two successive numbers).

Being bigger than every prime it can't be prime itself, yet it must factorise into primes and so is divisible by at least one prime number.

This is a contradiction and so there are infinitely many prime numbers.

INTERLUDE: STORY

Pam and the Prime Minister

The pure voices of the boy sopranos floated up to the lofty recesses of St Mersennes. "... primes without end ... a-men". The service was over.

Elisabeth turned to her friend and said, "Isn't he a dream – those eyes!"

But Pamela said, "I was more interested in his sermon. It seemed very persuasive but I'm sorry, Elisabeth, I'm still an agnostic. I just can't believe in your doctrine of the Infinitude of Primes. I mean, perhaps it *is* true that there are infinitely many prime numbers. I can't see how you could ever know for sure".

"But Pam, you can see here in *Primes Ancient and Modern* and here in *The Book of Common Primes* that there *are* primes for ever and ever into eternity. Look there's no sign of them petering out."

They had reached the church door and the young curate held out his hand.

"I trust you enjoyed the service, Elisabeth?" He greeted them, while looking at Pamela with his penetrating blue eyes.

"Oh yes", gushed Elisabeth, "I found the primes *so* inspiring. But I'm afraid my friend here is an unbeliever".

Pamela smiled sheepishly. "It's just that I can't see how you can be so certain. I admit that it seems very unlikely that the list of primes will ever come to an abrupt halt but ... I mean ... it *is* possible. After all primes become

rarer as you go among the larger numbers. Is it inconceivable that they eventually give out altogether?"

She pointed vaguely in the direction of the churchyard, but there were too many people behind them waiting to shake the curate's hand to continue the conversation.

"How about if you and Elisabeth come to the rectory next Sunday afternoon? We could talk some more over tea and scones."

* * * * *

Reverend Matthews poured the tea and passed round the excellent scones that Mrs Duffy had made.

"I'm sure Pam would like to believe that there are infinitely many primes but she doesn't seem to have enough faith."

"If only there was some way you could prove it to me," sighed Pamela, "but of course that's impossible. Even if I spent from now till the end of the world factorising numbers I'd only be considering a finite number of possibilities. There's no way the question can ever be settled."

"Well," said the curate, "you *do* believe that there are infinitely many numbers altogether don't you?"

"Oh yes, of course, that's obvious. I mean you just keep on adding one to get bigger and bigger numbers."

"And if I claimed that there was a biggest number?" he asked.

“Then I’d say what about that number plus one?”

“Exactly. I’d be forced to admit that my claim was false.”

“But that wouldn’t work for prime numbers,” protested Pam, “because all primes are odd ... except for the number two of course. And so the largest-prime-plus-one would be an even number so it couldn’t be prime. And the next number after that mightn’t be prime either.”

At this Reverend Matthews took a handful of cubes from the sugar bowl and laid them neatly in a row on the damask tablecloth. “Suppose,” he said, “that each of these sugar cubes represents a prime number. Here’s two and three, five, seven, eleven and thirteen, seventeen, nineteen and twenty-three. Now just suppose, for argument’s sake, that there *does* exist a largest prime.”

He scooped up the glistening white cubes and put them back into the sugar bowl. “Just suppose that this bowl contains every prime number up to the largest prime.”

“Well, alright then,” agreed Pamela, “just for the sake of argument. But don’t forget that I maintain that believing in a largest prime is just as illogical as believing in the Infinitude of Primes. You’d need infinite time to prove it one way or the other.”

“I hope it won’t take *that* long,” he said looking at his watch and smiling, “I have to conduct Evensong at six o'clock!” He picked up the pot containing all-the-primess-in-the-world and said, looking earnestly at Pam, “we have here every prime number that exists and, we’re supposing, there are only finitely many of them.”



“But a very large finite number,” said Elisabeth helpfully.

“Now we can multiply all these numbers together to get an exceedingly large number.”

“What if there’s not enough paper in the whole world to write it down?” asked Elisabeth.

“That’s of no consequence”, he assured her, “we can conceive and discuss numbers bigger than the number of atoms in the cosmos. Don’t forget, a number’s existence doesn’t depend on the vital statistics of our universe.”

“But I don’t see what you’re getting at”, said Pam as she took another scone. “The result of multiplying all the prime numbers won’t be a prime number itself, so where’s the contradiction, if there is one?”

“But would you agree that this product-of-all-primess will be divisible, exactly, by all prime numbers?”

“Yes Pam, don’t you see,” said Elisabeth excitedly, “every prime number will go into it exactly because every prime will be one of its factors!”

Pam did see. She was more concerned about where the argument would go from there.

“Well the product-of-all-primes will be divisible by all primes so the product-of-all-primes plus one can’t be divisible by *any* prime.” Reverend Matthews leaned over towards Pamela to make sure she got the point.

“You mean because no two consecutive numbers can have a common factor?”, said Pam thoughtfully.

“Exactly. So we’re brought to a number which has *no* prime factors. Now this product-of-all-primes-plus-one is too big to be a prime itself.”



He put his hand on Pam’s head to steady her from the impact of the contradiction that was about to follow. “But every number, if not prime itself, can be factorised into prime factors, so it *must* be divisible by at least one prime and hence we reach a contradiction. And remember Pam that contradiction only came about because we were foolish enough to contemplate a largest prime.”

Pam appeared to recover quickly from the shock of the contradiction, if she felt it at all. But in case of an aftershock his hand across the table steadied her arm.

Pam, in fact, was so deep in thought that she forgot for a moment that she even had an arm. She screwed up her pretty, little nose, trying to make sense of it all. At last she discovered the arm, drew it away from the young

curate's grasp and picked up an unused sugar cube which had lain unnoticed on her saucer.

"Well all that means," she said, "is that since it isn't divisible by any of the primes already in our pot it must itself be prime, one we overlooked. So we just pop this extra prime in the pot." And she dropped the sugar cube into the bowl.

"But", protested Reverend Matthews, "you'll just get the same contradiction all over again."

Pamela picked up a handful of the cubes and dropped them one by one back into the bowl. "So, as fast as you keep getting a contradiction I just keep adding more and more primes to the pot. I can always keep one step ahead of you." She grinned, confident that she had him beaten.

But Reverend Matthews yielded no ground. In fact he must have been about to deliver another intellectual earthquake because he felt the need to steady her arm again. "The point is Pam, you agreed that we had *all* primes in our finite pot, and now that I contradict you, you want to add another. That's hardly fair."

But Elisabeth came to her rescue. "Is it such a sin to change one's mind?"

"Look if it were a game of chess I'd be only too glad to let her change her mind to correct an oversight. But she can't claim to be always one step ahead of me just because I let her keep changing her move every time she lands in trouble. Besides, finding a prime that's not in the prime pot isn't an oversight. We agreed to *define* the

contents of the pot to consist of all primes. It's just an inescapable contradiction. And any assumption that leads to a contradiction must be false. Q.E.D.”

He appeared to think that this final blow would require a little extra support and it didn't matter that he upset the sugar bowl in the process because those little crystal cubes had served their purpose. Pamela displayed her discomfort at her intellectual position by blushing brightly. She looked down at the spilt cubes on the table as if pleading with them to deliver her the inescapable conclusion of the argument.

At last she looked up into Reverend Matthews' deep blue eyes and sighed, “I suppose you're right.”

But Elisabeth, who had becoming more and more agitated while all this was going on, said tersely, “I'm not so sure now. If believing in the Infinitude of Primes stops people from changing their minds I think I'd rather be an agnostic!”

